WILLIAMS' DECOMPOSITION OF THE LÉVY CONTINUUM RANDOM TREE AND SIMULTANEOUS EXTINCTION PROBABILITY FOR POPULATIONS WITH NEUTRAL MUTATIONS

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Abstract. We consider an initial Eve-population and a population of neutral mutants, such that the total population dies out in finite time. We describe the evolution of the Eve-population and the total population with continuous state branching processes, and the neutral mutation procedure can be seen as an immigration process with intensity proportional to the size of the population. First we establish a Williams' decomposition of the genealogy of the total population given by a continuum random tree, according to the ancestral lineage of the last individual alive. This allows us to give a closed formula for the probability of simultaneous extinction of the Eve-population and the total population.

1. Introduction

We consider an initial Eve-population whose size evolves as a continuous state branching process (CB), $Y^0 = (Y^0_t, t \geq 0)$, with branching mechanism ψ_{Eve}. We assume this population gives birth to a population of irreversible mutants. The new mutants population can be seen as an immigration process with rate proportional to the size of the Eve-population. We assume the mutations are neutral, so that this second population evolves according to the same branching mechanism as the Eve-population. This population of mutants gives birth also to a population of other irreversible mutants, with rate proportional to its size, and so on. In [2], we proved that the distribution of the total population size $Y = (Y_t, t \geq 0)$, which is a CB with immigration (CBI) proportional to its own size, is in fact a CB, whose branching mechanism ψ depends on the immigration intensity. The joint law of (Y^0, Y) is characterized by its Laplace transform, see Section 4.1.4. This model can also be viewed as a special case of multitype CB, with two types 0 and 1, the individuals of type 0 giving birth to offsprings of type 0 or 1, whereas individuals of type 1 only have type 1 offsprings, see [13, 6] for recent related works.

In the particular case of Y being a sub-critical or critical CB with quadratic branching mechanism ($\psi(u) = \alpha_0 u + \beta u^2$, $\beta > 0$, $\alpha_0 \geq 0$), the probability for the Eve-population to disappear at the same time as the whole population is known, see [17] for the critical case, $\alpha_0 = 0$, or Section 5 in [2] for the sub-critical case, $\alpha_0 > 0$. Our aim is to extend those results for the large class of CB with unbounded total variation and a.s. extinction. Formulas given in [2] could certainly be extended to a general branching mechanism, but first computations seem to be rather involved.

In fact, to compute those quantities, we choose here to rely on the description of the genealogy of sub-critical or critical CB introduced by Le Gall and Le Jan [12] and developed later by Duquesne and Le Gall [7], see also Lambert [10] for the genealogy of CBI with

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constant immigration rate. Le Gall and Le Jan defined via a Lévy process X the so-called height process $H = (H_t, t \geq 0)$ which codes a continuum random tree (CRT) that describes the genealogy of the CB (see the next section for the definition of H and the coding of the CRT). Initially, the CRT was introduced by Aldous [4] in the quadratic case: $\psi(\lambda) = \lambda^2$. Except in this quadratic case, the height process H is not Markov and so is difficult to handle. That is why they also introduce a measure-valued Markov process $(\rho_t, t \geq 0)$ called the exploration process and such that the closed support of the measure ρ_t is $[0, H_t]$ (see also the next section for the definition of the exploration process).

We shall be interested in the case where a.s. the extinction of the whole population holds in finite time. The branching mechanism of the total population, Y, is given by: for $\lambda \geq 0$,

$$\psi(\lambda) = \alpha_0\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(d\ell) \left(\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell \right), \tag{1}$$

where $\alpha_0 \geq 0$, $\beta \geq 0$ and π is a Radon measure on $(0, \infty)$ such that $\int_{(0,\infty)} (\ell \wedge \ell^2)\, \pi(d\ell) < \infty$. We shall assume that Y is of infinite variation, that is $\beta > 0$ or $\int_{(0,1)} \ell\pi(d\ell) = \infty$. We shall assume that a.s. the extinction of Y in finite time holds, that is, see Corollary 1.4.2 in [7], we assume that

$$\int^{+\infty} \frac{dv}{\psi(v)} < \infty. \tag{2}$$

We suppose that the process Y is the canonical process on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ of càdlàg paths and that the pair (Y, Y^0) is the canonical process on the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)^2$. Let P_x denote the law of the pair (Y, Y^0) (see [2]) started at $(Y_0, Y_0^0) = (x, x)$. The probability measure P_x is infinitely divisible and hence admits a canonical measure N: it is a σ-finite measure on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)^2$ such that

$$(Y, Y^0) \overset{(d)}{=} \sum_{i \in I} (Y^i, Y^{0,i})$$

where $((Y^i, Y^{0,i}), i \in I)$ are the atoms of a Poisson measure on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)^2$ with intensity $x\mathrm{N}(dY, dY^0)$. In particular, we have

$$\mathrm{E}_x[\mathrm{e}^{-\lambda Y_t}] = \exp(-x\mathrm{N}[1 - \mathrm{e}^{-\lambda Y_t}]) \tag{3}$$

and $u(\lambda, t) = \mathrm{N}[1 - \mathrm{e}^{-\lambda Y_t}]$ is the unique non-negative solution of

$$\int_{u(\lambda,t)}^{\lambda} \frac{dv}{\psi(v)} = t, \quad \text{for } t \geq 0 \text{ and } \lambda \geq 0. \tag{4}$$

Let $\tau_Y = \inf\{t > 0; Y_t = 0\}$ be the extinction time of Y. Letting λ go to ∞ in the previous equalities leads to

$$\mathrm{P}_x(\tau_Y < t) = \exp -x\mathrm{N}[\tau_Y \geq t],$$

where the positive function $c(t) = \mathrm{N}[\tau_Y \geq t]$ solves

$$\int_{c(t)}^{\infty} \frac{dv}{\psi(v)} = t, \quad \text{for } t > 0. \tag{5}$$

Let us consider the exploration process $(\rho_t, t \geq 0)$ associated with this CB. We denote by $\mathbb{N}$ its excursion measure. Recall that the closed support of the measure ρ_t is $[0, H_t]$, where H is the height process. Let L^a be the total local time at level a of the height process H (well-defined under $\mathbb{N}$). Then, the process $(L^a, a \geq 0)$ under $\mathbb{N}$ has the same distribution as the CB Y under N.

We decompose the exploration process, under the excursion measure, according to the maximum of the height process. In terms of the CRT, this means that we consider the longest rooted branch of the CRT and describe how the different subtrees are grafted along that branch, see Theorem 3.3. When the branching mechanism is quadratic, the height process H is a Brownian excursion and the exploration process ρ_t is, up to a constant, the Lebesgue measure on $[0, H_t]$. In that case, this decomposition corresponds to Williams' original decomposition of the Brownian excursion (see [18]). This kind of tree decomposition with respect to a particular branch (or a particular subtree) is not new, let us cite [9, 14] for instance, or [16, 15, 8] for related works on superprocesses.

We present in the introduction a Poisson decomposition for the CB only, and we refer to Theorem 3.3 for the decomposition of the exploration process. Conditionally on the extinction time τ_Y equal to m, we can represent the process Y as the sum of the descendants of the ancestors of the last individual alive. More precisely, let $\mathcal{N}'(d\ell, dt) = \sum_{i\in I} \delta_{(\ell_i,t_i)}(d\ell, dt)$ be a Poisson point measure with intensity

$$\mathbf{1}_{[0,m)}(t)\, \mathrm{e}^{-\ell c(m-t)}\, \ell\pi(d\ell)dt,$$

and

$$\kappa_{\max}(dt) = \sum_{i\in I} \ell_i \delta_{t_i}(dt) + 2\beta \mathbf{1}_{[0,m)}(t)dt. \tag{6}$$

Let $\mathrm{N}_t(dY)$ denote the law of $(Y(s-t), s \geq t)$ under N and $\sum_{j\in J} \delta_{(t_j,Y^j)}$ be, conditionally on $\mathcal{N}'$, a Poisson point measure with intensity

$$\kappa_{\max}(dt)\mathrm{N}_t[dY, \mathbf{1}_{\{\tau_Y \leq m\}}]$$

where $\mathrm{N}_t[dY, \mathbf{1}_{\{\tau_Y \leq m\}}]$ denotes the restriction of the measure N_t to the event $\{\tau_Y \leq m\}$.

The next result is a direct consequence of Theorem 3.3.

Proposition 1.1. *The process $\sum_{j\in J} Y^j$ is distributed as Y under N, conditionally on $\{\tau_Y = m\}$.*

Let $\tau_{Y^0} = \inf\{t > 0; Y^0_t = 0\}$ be the extinction time of the Eve-population. In the particular case where the branching mechanism of the Eve-population is given by a shift of ψ:

$$\psi_{\mathrm{Eve}}(\cdot) = \psi(\theta + \cdot) - \psi(\theta), \tag{7}$$

for some $\theta > 0$ and $\beta = 0$, the pruning procedure developed in [1] gives that the nodes of width ℓ_i correspond to a mutation with probability $1 - \mathrm{e}^{-\theta\ell_i}$. As $\beta = 0$ there is no mutation on the skeleton of the CRT outside the nodes. In particular, we see simultaneous extinction of the whole population and the Eve-population if there is no mutation on the nodes in the ancestral lineage of the last individual alive. This happens, conditionally on $\kappa_{\max}$, with probability

$$\mathrm{e}^{-\theta\sum_{i\in I} \ell_i}.$$

Integrating w.r.t. the law of $\mathcal{N}$ gives that the probability of simultaneous extinction, conditionally on $\{\tau_Y = m\}$, is under N, given by

$$\begin{aligned}
\mathrm{N}[\tau_{Y^0} = m|\tau_Y = m] &= \exp - \int \mathbf{1}_{[0,m)}(t)\, \mathrm{e}^{-\ell c(m-t)}\, \ell\pi(d\ell)dt \left[1 - \mathrm{e}^{-\theta\ell}\right] \\
&= \exp - \int_0^m [\psi'(c(m-t)+\theta) - \psi'(c(m-t))]\, dt \\
&= \exp - \int_0^m \phi'(c(t))\, dt,
\end{aligned}$$

where $\phi = \psi_{\mathrm{Eve}} - \psi$. Now, using that the distribution of (Y^0, Y) is infinitely divisible with canonical measure N, standard computations for Poisson measure yield that $\mathrm{P}_x(\tau_{Y^0} = m|\tau_Y = m) = \mathrm{N}[\tau_{Y^0} = m|\tau_Y = m]$ that is

$$\mathrm{P}_x(\tau_{Y^0} = m|\tau_Y = m) = \exp - \int_0^m \phi'(c(t))\, dt.$$

Notice that this formula is also valid for the quadratic branching mechanism ($\psi(u) = \alpha_0 u + \beta u^2$, $\beta > 0$, $\alpha_0 \geq 0$), see Remark 5.3 in [2].

In fact this formula is true in a general framework. Following [2], we consider the branching mechanisms of the total population and Eve-population are given by

$$\psi(\lambda) = \alpha_0\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(d\ell)[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell],$$

$$\psi_{\mathrm{Eve}}(\lambda) = \alpha_{\mathrm{Eve}}\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi_{\mathrm{Eve}}(d\ell)[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell],$$

and the immigration function

$$\phi(\lambda) = \psi_{\mathrm{Eve}}(\lambda) - \psi(\lambda) = \alpha_{\mathrm{Imm}}\lambda + \int_{(0,\infty)} \nu(d\ell)(1 - \mathrm{e}^{-\lambda\ell}),$$

where $\alpha_{\mathrm{Imm}} = \alpha_{\mathrm{Eve}} - \alpha_0 - \int_{(0,\infty)} \ell\nu(d\ell) \geq 0$ and $\pi = \pi_{\mathrm{Eve}} + \nu$, where π_{Eve} and ν are Radon measures on $(0,\infty)$ with $\int_{(0,\infty)} \ell\nu(d\ell) < \infty$. Notice the condition $\int_{(0,\infty)} \ell\nu(d\ell) < \infty$ is stronger than the usual condition on the immigration measure, $\int_{(0,\infty)}(1 \wedge \ell)\, \nu(d\ell) < \infty$, but is implied by the requirement that $\int_{(1,\infty)} \ell\nu(d\ell) < \int_{(1,\infty)} \ell\pi(d\ell) < \infty$.

Inspired by Theorem 3.3, we consider $\mathcal{N}(d\ell, dt, dz) = \sum_{i\in I} \delta_{(\ell_i,t_i,z_i)}(d\ell, dt, dz)$ a Poisson point measure with intensity

$$\mathbf{1}_{[0,m)}(t)\, \mathrm{e}^{-\ell c(m-t)}\, \ell\, [\pi_{\mathrm{Eve}}(d\ell)\delta_0(dz) + \nu(d\ell)\delta_1(dz)]\, dt. \tag{8}$$

Intuitively, the mark z_i indicates if the ancestor (of the last individual alive) alive at time t_i had a new mutation ($z_i = 1$) or not ($z_i = 0$). Note however that if $\beta > 0$ we have to take into account mutation on the skeleton. More precisely, let $T_1 = \min\{t_i, z_i = 1\}$ be the first mutation on the nodes in the ancestral lineage of the last individual alive and let T_2 be an exponential random time with parameter α_{Imm} independent of $\mathcal{N}$. The time T_2 corresponds to the first mutation on the skeleton for the ancestral lineage of the last individual alive. We set

$$\begin{cases} T_0 = \min(T_1, T_2) & \text{if } \min(T_1, T_2) < m, \\ T_0 = +\infty & \text{otherwise.} \end{cases} \tag{9}$$

In particular there is simultaneous extinction if and only if $T_0 = +\infty$.

For $t \geq 0$, let us denote by $\mathrm{N}_t(dY^0, dY)$ the joint law of $((Y^0(s-t), Y(s-t)), s \geq t)$ under N. Recall $\kappa_{\max}$ given by (6). Conditionally on $\mathcal{N}$ and T_2, let $\sum_{j \in J} \delta_{(t_j, Y^{0,j}, Y^j)}$ be a Poisson point measure, with intensity

$$\kappa_{\max}(dt)\mathrm{N}_t[(dY^0, dY),\ \mathbf{1}_{\{\tau_Y \leq m\}}].$$

We set

$$(Y'^0, Y') = \sum_{t_j < T_0} (Y^{0,j}, Y^j) + \sum_{t_j \geq T_0} (0, Y^j). \tag{10}$$

We write $\mathbb{Q}_m$ for the law of (Y'^0, Y') computed for a given value of m.

Theorem 1.2. *Under $\mathbb{Q}_m$, (Y'^0, Y') is distributed as (Y^0, Y) under $\mathrm{N}[\cdot|\tau_Y = m]$, or equivalently, under $\int_0^{+\infty} |c'(m)| \mathbb{Q}_m(\cdot) dm$, (Y'^0, Y') is distributed as (Y^0, Y) under* N.

Let us remark that this Theorem is very close to Theorem 3.3 but only deals with CB and does not specify the underlying genealogical structure. This is the purpose of a forthcoming paper [3] where the genealogy of multi-type CB is described.

Intuitively, conditionally on the last individual alive being at time m, until the first mutation in the ancestral lineage (that is for $t_j < T_0$) , its ancestors give birth to a population with initial Eve type which has to die before time m, and after the first mutation on the ancestral lineage (that is for $t_j \geq T_0$), there is no Eve-population in the descendants which still have to die before time m.

Now, using that the distribution of (Y^0, Y) is infinitely divisible with canonical measure N, standard computations for Poisson measure yield that $\mathrm{P}_x(\tau_{Y^0} = m|\tau_Y = m) = \mathrm{N}[\tau_{Y^0} = m|\tau_Y = m]$. As

$$\begin{aligned}
\mathrm{N}[\tau_{Y^0} = m|\tau_Y = m] &= \mathbb{Q}_m(T_0 = +\infty) \\
&= \mathbb{Q}_m(T_1 = +\infty)\mathbb{Q}_m(T_2 \geq m) \\
&= \mathrm{e}^{-\int_0^m dt \int_{(0,\infty)} \mathrm{e}^{-\ell c(m-t)}\, \ell \nu(d\ell)}\, \mathrm{e}^{-\alpha_{\mathrm{Imm}} m} \\
&= \mathrm{e}^{-\int_0^m dt\, \phi'(c(t))},
\end{aligned}$$

we deduce the following Corollary.

Corollary 1.3 (Probability of simultaneous extinction)**.** *We have for almost every $m > 0$*

$$\mathrm{P}_x(\tau_{Y^0} = m|\tau_Y = m) = \exp - \int_0^m \phi'(c(t))\, dt,$$

where c is the unique (non-negative) solution of (5).

The paper is organized as follows. In Section 2, we recall some facts on the genealogy of the CRT associated with a Lévy process. We prove a Williams' decomposition for the exploration process associated with the CRT in Section 3. We prove Theorem 1.2 in Section 4. Notice that Proposition 1.1 is a direct consequence of Theorem 1.2.

2. Notations

We recall here the construction of the Lévy continuum random tree (CRT) introduced in [12, 11] and developed later in [7]. We will emphasize on the height process and the exploration process which are the key tools to handle this tree. The results of this section are mainly extracted from [7].

2.1. **The underlying Lévy process.** We consider a $\mathbb{R}$-valued Lévy process $(X_t, t \geq 0)$ with Laplace exponent ψ (for $\lambda \geq 0$ $\mathbb{E}\left[e^{-\lambda X_t}\right] = e^{t\psi(\lambda)}$) satisfying (1) and (2). Let $I = (I_t, t \geq 0)$ be the infimum process of X, $I_t = \inf_{0 \leq s \leq t} X_s$, and let $S = (S_t, t \geq 0)$ be the supremum process, $S_t = \sup_{0 \leq s \leq t} X_s$. We will also consider for every $0 \leq s \leq t$ the infimum of X over $[s, t]$:

$$I_t^s = \inf_{s \leq r \leq t} X_r.$$

The point 0 is regular for the Markov process $X - I$, and $-I$ is the local time of $X - I$ at 0 (see [5], chap. VII). Let $\mathbb{N}$ be the associated excursion measure of the process $X - I$ away from 0, and $\sigma = \inf\{t > 0; X_t - I_t = 0\}$ the length of the excursion of $X - I$ under $\mathbb{N}$. We will assume that under $\mathbb{N}$, $X_0 = I_0 = 0$.

Since X is of infinite variation, 0 is also regular for the Markov process $S - X$. The local time, $L = (L_t, t \geq 0)$, of $S - X$ at 0 will be normalized so that

$$\mathbb{E}[e^{-\beta S_{L_t^{-1}}}] = e^{-t\psi(\beta)/\beta},$$

where $L_t^{-1} = \inf\{s \geq 0; L_s \geq t\}$ (see also [5] Theorem VII.4 (ii)).

2.2. **The height process and the Lévy CRT.** For each $t \geq 0$, we consider the reversed process at time t, $\hat{X}^{(t)} = (\hat{X}_s^{(t)}, 0 \leq s \leq t)$ by:

$$\hat{X}_s^{(t)} = X_t - X_{(t-s)-} \quad \text{if} \quad 0 \leq s < t,$$

and $\hat{X}_t^{(t)} = X_t$. The two processes $(\hat{X}_s^{(t)}, 0 \leq s \leq t)$ and $(X_s, 0 \leq s \leq t)$ have the same law. Let $\hat{S}^{(t)}$ be the supremum process of $\hat{X}^{(t)}$ and $\hat{L}^{(t)}$ be the local time at 0 of $\hat{S}^{(t)} - \hat{X}^{(t)}$ with the same normalization as L.

Definition 2.1 ([7], Definition 1.2.1 and Theorem 1.4.3)**.** *There exists a process $H = (H_t, t \geq 0)$, called the height process, such that for all $t \geq 0$, a.s. $H_t = \hat{L}_t^{(t)}$, and $H_0 = 0$. Because of hypothesis (2), the height process H is continuous.*

The height process $(H_t, t \in [0, \sigma])$ under $\mathbb{N}$ codes a continuous genealogical structure, the Lévy CRT, via the following procedure.

(i) To each $t \in [0, \sigma]$ corresponds a vertex at generation H_t.
(ii) Vertex t is an ancestor of vertex t' if $H_t = H_{[t,t']}$, where

$$H_{[t,t']} = \inf\{H_u, u \in [t \wedge t', t \vee t']\}. \tag{11}$$

In general $H_{[t,t']}$ is the generation of the last common ancestor to t and t'.
(iii) We put $d(t, t') = H_t + H_{t'} - 2H_{[t,t']}$ and identify t and t' ($t \sim t'$) if $d(t, t') = 0$.

The Lévy CRT coded by H is then the quotient set $[0, \sigma]/\sim$, equipped with the distance d and the genealogical relation specified in (ii).

Let $(\tau_s, s \geq 0)$ be the right continuous inverse of $-I$: $\tau_s = \inf\{t > 0; -I_t > s\}$. Recall that $-I$ is the local time of $X - I$ at 0. Let L_t^a denote the local time at level a of H until time t, see Section 1.3 in [7].

Theorem 2.2 ([7], Theorem 1.4.1)**.** *The process $(L_{\tau_x}^a, a \geq 0)$ is under $\mathbb{P}$ (resp. $\mathbb{N}$) defined as Y under P_x (resp. N).*

In what follows, we will use the notation $\mathbb{N}$ instead of N for the excursion measure to stress that we consider the genealogical structure of the branching process.

2.3. **The exploration process.** The height process is not Markov. But it is a simple function of a measure-valued Markov process, the so-called exploration process.

If E is a Polish space, let $\mathcal{B}(E)$ (resp. $\mathcal{B}_+(E)$) be the set of real-valued measurable (resp. and non-negative) functions defined on E endowed with its Borel σ-field, and let $\mathcal{M}(E)$ (resp. $\mathcal{M}_f(E)$) be the set of σ-finite (resp. finite) measures on E, endowed with the topology of vague (resp. weak) convergence. For any measure $\mu \in \mathcal{M}(E)$ and $f \in \mathcal{B}_+(E)$, we write

$$\langle \mu, f\rangle = \int f(x)\,\mu(dx).$$

The exploration process $\rho = (\rho_t, t \geq 0)$ is a $\mathcal{M}_f(\mathbb{R}_+)$-valued process defined as follows: for every $f \in \mathcal{B}_+(\mathbb{R}_+)$,

$$\langle \rho_t, f\rangle = \int_{[0,t]} d_s I_t^s f(H_s),$$

or equivalently

$$\rho_t(dr) = \sum_{\substack{0<s\leq t\\ X_{s-}<I_t^s}} (I_t^s - X_{s-})\delta_{H_s}(dr) + \beta \mathbf{1}_{[0,H_t]}(r)dr. \tag{12}$$

In particular, the total mass of ρ_t is $\langle \rho_t, 1\rangle = X_t - I_t$.

For $\mu \in \mathcal{M}(\mathbb{R}_+)$, we set

$$H(\mu) = \sup \operatorname{Supp} \mu, \tag{13}$$

where $\operatorname{Supp} \mu$ is the closed support of μ, with the convention $H(0) = 0$. We have

Proposition 2.3 ([7], Lemma 1.2.2)**.** *Almost surely, for every $t > 0$,*

- $H(\rho_t) = H_t$,
- $\rho_t = 0$ *if and only if* $H_t = 0$,
- *if* $\rho_t \neq 0$, *then* $\operatorname{Supp} \rho_t = [0, H_t]$.

In the definition of the exploration process, as X starts from 0, we have $\rho_0 = 0$ a.s. To state the Markov property of ρ, we must first define the process ρ started at any initial measure $\mu \in \mathcal{M}_f(\mathbb{R}_+)$.

For $a \in [0, \langle \mu, 1\rangle]$, we define the erased measure $k_a\mu$ by

$$k_a\mu([0, r]) = \mu([0, r]) \wedge (\langle \mu, 1\rangle - a), \quad \text{for } r \geq 0.$$

If $a > \langle \mu, 1\rangle$, we set $k_a\mu = 0$. In other words, the measure $k_a\mu$ is the measure μ erased by a mass a from the top of $[0, H(\mu)]$.

For $\nu, \mu \in \mathcal{M}_f(\mathbb{R}_+)$, and μ with compact support, we define the concatenation $[\mu, \nu] \in \mathcal{M}_f(\mathbb{R}_+)$ of the two measures by:

$$\big\langle [\mu, \nu], f\big\rangle = \big\langle \mu, f\big\rangle + \big\langle \nu, f(H(\mu) + \cdot)\big\rangle, \quad f \in \mathcal{B}_+(\mathbb{R}_+).$$

Finally, we set for every $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ and every $t > 0$ $\rho_t^\mu = \big[k_{-I_t}\mu, \rho_t\big]$. We say that $(\rho_t^\mu, t \geq 0)$ is the process ρ started at $\rho_0^\mu = \mu$, and write $\mathbb{P}_\mu$ for its law. Unless there is an ambiguity, we shall write ρ_t for ρ_t^μ.

Proposition 2.4 ([7], Proposition 1.2.3)**.** *The process $(\rho_t, t \geq 0)$ is a càd-làg strong Markov process in $\mathcal{M}_f(\mathbb{R}_+)$.*

Notice that $\mathbb{N}$ is also the excursion measure of the process ρ away from 0, and that σ, the length of the excursion, is $\mathbb{N}$-a.e. equal to $\inf\{t > 0; \rho_t = 0\}$.

2.4. **The dual process and representation formula.** We shall need the $\mathcal{M}_f(\mathbb{R}_+)$-valued process $\eta = (\eta_t, t \geq 0)$ defined by

$$\eta_t(dr) = \sum_{\substack{0<s\leq t\\ X_{s-}<I_t^s}} (X_s - I_t^s)\delta_{H_s}(dr) + \beta \mathbf{1}_{[0,H_t]}(r)dr.$$

The process η is the dual process of ρ under $\mathbb{N}$ thanks to the following time reversal property: recall σ denotes the length of the excursion under $\mathbb{N}$.

Proposition 2.5 ([7], Corollary 3.1.6)**.** *The processes* $((\rho_s, \eta_s); s \geq 0)$ *and* $((\eta_{(\sigma-s)-}, \rho_{(\sigma-s)-}); s \geq 0)$ *have the same distribution under* $\mathbb{N}$*.*

It also enjoys the snake property: for all $t \geq 0, s \geq 0$

$$(\rho_t, \eta_t)_{[0,H_{[t,s]})} = (\rho_s, \eta_s)_{[0,H_{[t,s]})},$$

that is the measures ρ and η between two instants coincide up to the minimum of the height process between those two instants.

We recall the Poisson representation of (ρ, η) under $\mathbb{N}$. Let $\mathcal{N}_*(dx\,d\ell\,du)$ be a Poisson point measure on $[0, +\infty)^3$ with intensity

$$dx\,\ell\pi(d\ell)\mathbf{1}_{[0,1]}(u)du.$$

For every $a > 0$, let us denote by $\mathbb{M}_a$ the law of the pair (μ_a, ν_a) of finite measures on $\mathbb{R}_+$ defined by: for $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$\langle \mu_a, f\rangle = \int \mathcal{N}_*(dx\,d\ell\,du)\mathbf{1}_{[0,a]}(x)u\ell f(x),$$

$$\langle \nu_a, f\rangle = \int \mathcal{N}_*(dx\,d\ell\,du)\mathbf{1}_{[0,a]}(x)\ell(1-u) f(x).$$

We finally set $\mathbb{M} = \int_0^{+\infty} da\ \mathrm{e}^{-\alpha_0 a}\,\mathbb{M}_a$.

Proposition 2.6 ([7], Proposition 3.1.3)**.** *For every non-negative measurable function F on $\mathcal{M}_f(\mathbb{R}_+)^2$,*

$$\mathbb{N}\left[\int_0^\sigma F(\rho_t, \eta_t)\ dt\right] = \int \mathbb{M}(d\mu\,d\nu)F(\mu,\nu),$$

where $\sigma = \inf\{s > 0; \rho_s = 0\}$ denotes the length of the excursion.

We can then deduce the following Proposition.

Proposition 2.7. *For every non-negative measurable function F on $\mathcal{M}_f(\mathbb{R}_+)^2$,*

$$\mathbb{N}\left[\int_0^\sigma F(\rho_t, \eta_t)\ dL_t^a\right] = \mathrm{e}^{-\alpha_0 a}\int \mathbb{M}_a(d\mu\,d\nu)F(\mu,\nu),$$

where $\sigma = \inf\{s > 0; \rho_s = 0\}$ denotes the length of the excursion.

3. Williams' decomposition

We work under the excursion measure. As the height process is continuous, its supremum $H_{\max} = \sup\{H_r; r \in [0,\sigma]\}$ is attained. Let $T_{\max} = \inf\{s \geq 0; H_s = H_{\max}\}$.

For every $m > 0$, we set $T_m(\rho) = \inf\{s > 0, H_s(\rho) = m\}$ the first hitting time of m for the height process. When there is no need to stress the dependence in ρ, we shall write T_m for $T_m(\rho)$. Recall the function c defined by (5) is equal to

$$c(m) = \mathbb{N}[T_m < \infty] = \mathbb{N}[H_{\max} \leq m]. \tag{14}$$

We set $\rho_d = (\rho_{T_{\max}+s}, s \geq 0)$ and $\rho_g = (\rho_{(T_{\max}-s)+}, s \geq 0)$, where $x_+ = \max(x, 0)$.

For every finite measure with compact support μ, we write $\mathbb{P}^*_\mu$ for the law of the exploration process ρ starting at μ and killed when it first reaches 0. We also set

$$\hat{\mathbb{P}}^*_\mu := \lim_{\varepsilon \to 0} \mathbb{P}^*_\mu(\cdot \,|\, H(\mu) \leq H_{\max} \leq H(\mu) + \varepsilon).$$

We now describe the probability measure $\hat{\mathbb{P}}^*_\mu$ via a Poisson decomposition. Let (α_i, β_i), $i \in I$ be the excursion intervals of the process $X - I$ away from 0 (well defined under $\mathbb{P}^*_\mu$ or under $\hat{\mathbb{P}}^*_\mu$). For every $i \in I$, we define $h_i = H_{\alpha_i}$ and the measure-valued process ρ^i by the formula

$$\langle \rho^i_t, f \rangle = \int_{(h_i, +\infty)} f(x - h_i) \rho_{(\alpha_i + t) \wedge \beta_i}(dx).$$

We then have the following result.

Lemma 3.1. *Under the probability $\hat{\mathbb{P}}^*_\mu$, the point measure $\sum_{i \in I} \delta_{(h_i, \rho^i)}$ is a Poisson point measure with intensity $\mu(dr)\mathbb{N}[\cdot, H_{max} \leq m - r]$.*

Proof. We know (cf Lemma 4.2.4 of [7]) that the point measure $\sum_{i \in I} \delta_{(h_i, \rho^i)}$ is under $\mathbb{P}^*_\mu$ a Poisson point measure with intensity $\mu(dr)\mathbb{N}(d\rho)$. The result follows then easily from standard results on Poisson point measures. □

Remark 3.2. Lemma 3.1 gives also that, for every finite measure with compact support μ, if we write $\mu_a = \mu(\cdot \cap [0, a])$,

$$\hat{\mathbb{P}}^*_\mu = \lim_{a \to H(\mu)} \mathbb{P}^*_{\mu_a}(\cdot \,|\, H_{\max} \leq H(\mu)).$$

Theorem 3.3 (Williams' Decomposition)**.**

(i) *The law of H_{max} is characterized by $\mathbb{N}[H_{max} \leq m] = c(m)$, where c is the unique non-negative solution of (5).*

(ii) *Conditionally on $H_{max} = m$, the law of $(\rho_{T_{max}}, \eta_{T_{max}})$ is under $\mathbb{N}$ the law of*

$$\left(\sum_{i \in I} v_i r_i \delta_{t_i} + \beta \mathbf{1}_{[0,m]}(t) dt, \sum_{i \in I} (1 - v_i) r_i \delta_{t_i} + \beta \mathbf{1}_{[0,m]}(t) dt \right),$$

where $\sum \delta_{(v_i, r_i, t_i)}$ is a Poisson measure with intensity

$$\mathbf{1}_{[0,1]}(v) \mathbf{1}_{[0,m]}(t) \, \mathrm{e}^{-rc(m-t)} \, dv \, r\pi(dr) \, dt.$$

(iii) *Under $\mathbb{N}$, conditionally on $H_{max} = m$, and $(\rho_{T_{max}}, \eta_{T_{max}})$, (ρ_d, ρ_g) are independent and ρ_d (resp. ρ_g) is distributed as ρ (resp. η) under $\hat{\mathbb{P}}^*_{\rho_{T_{max}}}$ (resp. $\hat{\mathbb{P}}^*_{\eta_{T_{max}}}$).*

Notice (i) is a consequence of (14). Point (ii) is reminiscent of Theorem 4.6.2 in [7] which gives the description of the exploration process at a first hitting time of the Lévy snake.

The end of this section is devoted to the proof of (ii) and (iii) of this Theorem.

Let $m > a > 0$ be fixed. Let $\varepsilon > 0$. Recall $T_m = \inf\{t > 0; H_t = m\}$ is the first hitting time of m for the height process, and set $L_m = \sup\{t < \sigma; H_t = m\}$ for the last hitting time of m, with the convention that $\inf \emptyset = +\infty$ and $\sup \emptyset = +\infty$. We consider the minimum of H between T_m and L_m: $H_{[T_m, L_m]} = \min\{H_t; t \in [T_m, L_m]\}$.

We set $\rho^{(d)} = (\rho_{T_{\max,a}+t}, t \geq 0)$, with

$$T_{\max,a} = \inf\{t > T_{\max}, H_s = a\},$$

the path of the exploration process on the right of $T_{\max}$ after the hitting time of a, and $\rho^{(g)} = (\rho_{(L_{\max,a}-t)-}, t \geq 0)$, with $L_{\max,a} = \sup\{t < T_{\max}; H_t = a\}$, the returned path of the exploration process on the left of $T_{\max}$ before its last hitting time of a. Let us note that, by time reversal (see Proposition 2.5), the process $\rho^{(g)}$ is of the same type as η. This remark will be used later.

To prove the Theorem, we shall compute

$$A_0 = \mathbb{N}\left[F_1(\rho^{(g)})F_2(\rho^{(d)})F_3(\rho_{T_{\max}|[0,a]})F_4(\eta_{T_{\max}|[0,a]})\mathbf{1}_{\{m\leq H_{\max}<m+\varepsilon\}}\right]$$

and let ε go down to 0. We shall see in Lemma 3.4, that adding $\mathbf{1}_{\{H_{[T_m,L_m]}>a\}}$ in the integrand does not change the asymptotic behavior as ε goes down to 0. Intuitively, if the maximum of the height process is between m and $m+\varepsilon$, outside a set of small measure, the height process does not reach level a between the first and last hitting time of m. So that we shall compute first

$$A = \mathbb{N}\left[F_1(\rho^{(g)})F_2(\rho^{(d)})F_3(\rho_{T_{\max}|[0,a]})F_4(\eta_{T_{\max}|[0,a]})\mathbf{1}_{\{H_{[T_m,L_m]}>a, m\leq H_{\max}<m+\varepsilon\}}\right]. \tag{15}$$

Notice that on $\{H_{[T_m,L_m]} > a\}$, we have $T_{\max,a} = T_{m,a} := \inf\{s > T_m, H_s(\rho) = a\}$ and, from the snake property, $\rho_{T_{\max}|[0,a]} = \rho_{T_m|[0,a]}$ and $\eta_{T_{\max}|[0,a]} = \eta_{T_m|[0,a]}$, so that

$$A = \mathbb{N}\left[F_1(\rho^{(g)})F_2((\rho_{T_{m,a}+t}, t \geq 0))F_3(\rho_{T_m|[0,a]})F_4(\eta_{T_m|[0,a]})\mathbf{1}_{\{H_{[T_m,L_m]}>a, m\leq H_{\max}<m+\varepsilon\}}\right].$$

Let us remark that, we have

$$\mathbf{1}_{\{H_{[T_m,L_m]}>a, m\leq H_{\max}<m+\varepsilon\}} = \mathbf{1}_{\{m\leq \sup\{H_u, 0\leq u\leq T_{m,a}\}<m+\varepsilon\}}\mathbf{1}_{\{\sup\{H_u, u\geq T_{m,a}\}<m\}}.$$

By using the strong Markov property of the exploration process at time $T_{m,a}$, we get

$$A=\mathbb{N}\Big[F_1(\rho^{(g)})F_4(\eta_{T_m|[0,a]})\mathbf{1}_{\{m\leq \sup\{H_u, 0\leq u\leq T_{m,a}\}<m+\varepsilon\}}F_3(\rho_{T_m|[0,a]})\mathbb{E}^*_{\rho_{T_m|[0,a]}}\big[F_2(\rho)\mathbf{1}_{\{H_{\max}<m\}}\big]\Big]$$

and so, by conditioning, we get

$$A = \mathbb{N}\left[F_1(\rho^{(g)})F_4(\eta_{T_m|[0,a]})G_2(\rho_{T_m|[0,a]})\mathbf{1}_{\{H_{[T_m,L_m]}>a, m\leq H_{\max}<m+\varepsilon\}}\right],$$

where $G_2(\mu) = F_3(\mu)\mathbb{E}^*_\mu[F_2(\rho)|H_{\max} < m]$. Using time reversibility (see Proposition 2.5) and the strong Markov property at time $T_{m,a}$ again, we have

$$\begin{aligned} A &= \mathbb{N}\left[F_1(\rho^{(d)})F_4(\rho_{T_m|[0,a]})G_2(\eta_{T_m|[0,a]})\mathbf{1}_{\{H_{[T_m,L_m]}>a, m\leq H_{\max}<m+\varepsilon\}}\right] \\ &= \mathbb{N}\left[G_1(\rho_{T_m|[0,a]})G_2(\eta_{T_m|[0,a]})\mathbf{1}_{\{H_{[T_m,L_m]}>a, m\leq H_{\max}<m+\varepsilon\}}\right], \end{aligned}$$

where $G_1(\mu) = F_4(\mu)\mathbb{E}^*_\mu[F_1(\rho)|H_{\max} < m]$.

Now, we use ideas from the proof of Theorem 4.6.2 of [7]. Let us recall the excursion decomposition of the exploration process above level a. We set $\tau^a_s = \inf\left\{r, \int_0^r du\, \mathbf{1}_{\{H_u\leq a\}} > s\right\}$. Let $\mathcal{E}_a$ be the σ-field generated by the process $(\tilde{\rho}_s,\ s \geq 0) := (\rho_{\tau^a_s},\ s \geq 0)$. We also set $\tilde{\eta}_s = \eta_{\tau^a_s}$.

Let (α_i, β_i), $i \in I$ be the excursion intervals of H above level a. For every $i \in I$ we define the measure-valued process ρ^i by setting

$$\begin{cases} \langle \rho^i_s, \varphi\rangle = \int_{(a,+\infty)} \rho_{\alpha_i+s}(dr)\varphi(r-a) & \text{if } 0 < s < \beta_i - \alpha_i, \\ \rho_s = 0 & \text{if } s = 0 \text{ or } s \geq \beta_i - \alpha_i, \end{cases}$$

and the process η^i similarly. We also define the local time at the beginning of excursion ρ^i by $\ell_i = L^a_{\alpha_i}$. Then, under $\mathbb{N}$, conditionally on $\mathcal{E}_a$, the point measure

$$\sum_{i\in I} \delta_{(\ell_i, \rho^i, \eta^i)}$$

is a Poisson measure with intensity $\mathbf{1}_{[0,L^a_\sigma]}(\ell) d\ell \mathbb{N}[d\rho\, d\eta]$.

In particular, we have

$$A = \mathbb{N}\left[\sum_{i\in I}\prod_{j\neq i} \mathbf{1}_{\{T_m(\rho^j)=+\infty\}} G_1(\rho_{\alpha_i}) G_2(\eta_{\alpha_i}) \mathbf{1}_{\{m\leq H_{\max}(\rho^i)<m+\varepsilon\}}\right].$$

Let us denote by $(\tau^a_\ell, \ell \geq 0)$ the right-continuous inverse of $(L^a_s, s \geq 0)$. Palm formula for Poisson point measures yields

$$\begin{aligned} A &= \mathbb{N}\left[\mathbb{N}\left[\sum_{i\in I}\prod_{j\neq i} \mathbf{1}_{\{T_m(\rho^j)=+\infty\}} G_1(\rho_{\alpha_i}) G_2(\eta_{\alpha_i}) \mathbf{1}_{\{m\leq H_{\max}(\rho^i)<m+\varepsilon\}} \,\Big|\, \mathcal{E}_a\right]\right] \\ &= \mathbb{N}\left[\int_0^{L^a_\sigma} d\ell\, G_1(\rho_{\tau^a_\ell}) G_2(\eta_{\tau^a_\ell}) \mathbb{N}[m \leq H_{\max} < m+\varepsilon] \mathbb{N}\left[\prod_{j\in I} \mathbf{1}_{\{T_m(\rho^j)=+\infty\}} \,\Big|\, \mathcal{E}_a\right]\right]. \end{aligned}$$

A time-change then gives

$$A = v(m-a,\varepsilon)\mathbb{N}\left[\int_0^\sigma dL^a_s G_1(\rho_s) G_2(\eta_s)\, \mathrm{e}^{-c(m-a)L^a_\sigma}\right], \tag{16}$$

where $v(x, \varepsilon) = c(x) - c(x + \varepsilon) = \mathbb{N}[x \leq H_{\max} < x + \varepsilon]$. We have

$$\begin{aligned} A &= v(m-a,\varepsilon)\mathbb{N}\left[\int_0^\sigma dL^a_s G_1(\rho_s) G_2(\eta_s)\, \mathrm{e}^{-c(m-a)L^a_s}\, \mathrm{e}^{-c(m-a)(L^a_\sigma - L^a_s)}\right] \\ &= v(m-a,\varepsilon)\mathbb{N}\left[\int_0^\sigma dL^a_s G_1(\rho_s) G_2(\eta_s)\, \mathrm{e}^{-c(m-a)L^a_s}\, \mathrm{e}^{-\langle \rho_s, \mathbb{N}[1-\mathrm{e}^{-c(m-a)L^{a-\cdot}_\sigma}]\rangle}\right], \end{aligned}$$

where we used for the last equality that the predictable projection of $\mathrm{e}^{-\lambda(L^a_\sigma - L^a_s)}$ is given by $\mathrm{e}^{-\langle \rho_s, \mathbb{N}[1-\mathrm{e}^{-\lambda L^{a-\cdot}_\sigma}]\rangle}$. Notice that by using the excursion decomposition above level $0 < r < m$, we have

$$c(m) = \mathbb{N}[T_m < \infty] = \mathbb{N}[1 - \mathrm{e}^{-c(m-r)L^r_\sigma}].$$

In particular, we get

$$A = v(m-a,\varepsilon)\mathbb{N}\left[\int_0^\sigma dL^a_s G_1(\rho_s) G_2(\eta_s)\, \mathrm{e}^{-c(m-a)L^a_s}\, \mathrm{e}^{-\langle \rho_s, c(m-\cdot)\rangle}\right].$$

Using time reversibility, we have

$$A = v(m-a,\varepsilon)\mathbb{N}\left[\int_0^\sigma dL^a_s G_1(\eta_s) G_2(\rho_s)\, \mathrm{e}^{-c(m-a)(L^a_\sigma - L^a_s)}\, \mathrm{e}^{-\langle \eta_s, c(m-\cdot)\rangle}\right].$$

Similar computations as those previously done give

$$\begin{aligned} A &= v(m-a,\varepsilon)\mathbb{N}\left[\int_0^\sigma dL_s^a G_1(\eta_s)G_2(\rho_s)\,\mathrm{e}^{-\langle \eta_s+\rho_s, c(m-\cdot)\rangle}\right] \\ &= v(m-a,\varepsilon)\mathbb{N}\left[\int_0^\sigma dL_s^a G_1(\rho_s)G_2(\eta_s)\,\mathrm{e}^{-\langle \rho_s+\eta_s, c(m-\cdot)\rangle}\right]. \end{aligned}$$

Using Proposition 2.7, we get

$$A = v(m-a,\varepsilon)\,\mathrm{e}^{-\alpha_0 a}\int \mathbb{M}_a(d\mu\, d\nu)G_1(\mu)G_2(\nu)\,\mathrm{e}^{-\langle \mu+\nu, c(m-\cdot)\rangle}.$$

We can give a first consequence of the previous computation.

Lemma 3.4. *We have*

$$\mathbb{N}[H_{[T_m,L_m]} > a, m \le H_{max} < m+\varepsilon] = c'(m)\frac{c(m-a)-c(m-a+\varepsilon)}{c'(m-a)}.$$

Proof. Taking $F_1 = F_2 = F_3 = F_4 = 1$ in (16), we deduce that

$$\mathbb{N}[H_{[T_m,L_m]} > a, m \le H_{\max} < m+\varepsilon] = v(m-a,\varepsilon)\mathbb{N}\left[L_\sigma^a\,\mathrm{e}^{-c(m-a)L_\sigma^a}\right].$$

Let $a_0 > 0$ and let us compute $B(a_0,a) = \mathrm{N}\left[L_\sigma^a\,\mathrm{e}^{-c(a_0)L_\sigma^a}\right]$. Thanks to Theorem 2.2, notice that

$$B(a_0,a) = \mathrm{N}\left[Y_a\,\mathrm{e}^{-c(a_0)Y_a}\right] = \frac{\partial_{a_0}\mathrm{N}[1-\mathrm{e}^{-c(a_0)Y_a}]}{c'(a_0)}.$$

On the other hand, we have

$$c(a+a_0) = \mathrm{N}[Y_{a+a_0} > 0] = \mathrm{N}[1-\mathrm{E}_{Y_a}[Y_{a_0}=0]] = \mathrm{N}\left[1-\mathrm{e}^{-Y_a c(a_0)}\right],$$

where we used the Markov property of Y at time a under N for the second equality and (3) with λ going to infinity for the last. Thus, we get $B(a_0,a) = \dfrac{c'(a_0+a)}{c'(a_0)}$. We deduce that

$$\begin{aligned} \mathbb{N}[H_{[T_m,L_m]} > a, m \le H_{\max} < m+\varepsilon] &= v(m-a,\varepsilon)B(a-m,a) \\ &= c'(m)\frac{c(m-a)-c(m-a+\varepsilon)}{c'(m-a)}. \end{aligned}$$

□

Since F_1, F_2, F_3 and F_4 are bounded, say by C, we have $|A-A_0| \le C^4\mathbb{N}[H_{[T_m,L_m]} < a, m \le H_{\max} < m+\varepsilon]$. From Lemma 3.4, we deduce that

$$\lim_{\varepsilon\to 0}\frac{|A-A_0|}{\mathbb{N}[m \le H_{\max} < m+\varepsilon]} \le C^4\left[1-\lim_{\varepsilon\to 0}\frac{\mathbb{N}[H_{[T_m,L_m]} > a, m \le H_{\max} < m+\varepsilon]}{\mathbb{N}[m \le H_{\max} < m+\varepsilon]}\right] = 0.$$

We deduce that

$$\begin{aligned}
\lim_{\varepsilon\to 0} & \frac{\mathbb{N}\left[F_1(\rho^{(g)})F_2(\rho^{(d)})F_3(\rho_{T_{\max}|[0,a]})F_4(\eta_{T_{\max}|[0,a]})\mathbf{1}_{\{m\le H_{\max}<m+\varepsilon\}}\right]}{\mathbb{N}[m\le H_{\max}<m+\varepsilon]} \\
&= \frac{c'(m-a)}{c'(m)}\,\mathrm{e}^{-\alpha_0 a}\int \mathbb{M}_a(d\mu\,d\nu)G_1(\mu)G_2(\nu)\,\mathrm{e}^{-\langle \mu+\nu, c(m-\cdot)\rangle} \\
&= \frac{\int \mathbb{M}_a(d\mu\,d\nu)G_1(\mu)G_2(\nu)\,\mathrm{e}^{-\langle \mu+\nu, c(m-\cdot)\rangle}}{\int \mathbb{M}_a(d\mu\,d\nu)\,\mathrm{e}^{-\langle \mu+\nu, c(m-\cdot)\rangle}} \\
&= \int \tilde{\mathbb{M}}_a(d\mu\,d\nu)G_1(\mu)G_2(\nu) \\
&= \int \tilde{\mathbb{M}}_a(d\mu\,d\nu)F_4(\nu)\mathbb{E}^*_\nu[F_1(\rho^{(d)})|H_{\max}<m]F_3(\mu)\mathbb{E}^*_\mu[F_2(\rho^{(d)})|H_{\max}<m],
\end{aligned}$$

where

$$\begin{aligned}
\mu(dt) &= \sum_{i\in I} u_i\ell_i\delta_{t_i} + \beta\mathbf{1}_{[0,a]}(t)dt \\
\nu(dt) &= \sum_{i\in I} (1-u_i)\ell_i\delta_{t_i} + \beta\mathbf{1}_{[0,a]}(t)dt,
\end{aligned}$$

and $\sum_{i\in I}\delta_{(x_i,\ell_i,t_i)}$ is under $\tilde{\mathbb{M}}_a$ a Poisson point measure on $[0,+\infty)^3$ with intensity

$$\mathbf{1}_{[0,a]}(t)dt\,\ell\,\mathrm{e}^{-\ell c(m-t)}\,\pi(d\ell)\mathbf{1}_{[0,1]}(u)du.$$

Standard results on measure decomposition imply there exists a regular version of the probability measure $\mathbb{N}[\,\cdot\,|H_{\max}=m]$ and that, for almost every non-negative m,

$$\mathbb{N}[\,\cdot\,|H_{\max}=m] = \lim_{\varepsilon\to 0}\mathbb{N}[\,\cdot\,|m\le H_{\max}<m+\varepsilon].$$

This gives (ii) and (iii) of Theorem 3.3 since F_1, F_2, F_3, F_4 are arbitrary continuous functionals and by Remark 3.2.

4. Proof of Theorem 1.2

The proof of this Theorem relies on the computation of the Laplace transform for (Y'^0, Y') and is given in the next three paragraphs. The next paragraph gives some preliminary computations.

4.1. **Preliminary computations.**

4.1.1. *Law of* T_0. Recall the definition of $\mathbb{Q}_m$ as the law of (Y'^0, Y') defined by (10) and T_0 defined by (9) as the first mutation undergone by the last individual alive.

For $r<m$, we have

$$\begin{aligned}
\mathbb{Q}_m(T_0\in[r,r+dr], T_0=T_2) &= \mathbb{Q}_m(T_2\in[r,r+dr])\mathbb{Q}_m(T_1>r) \\
&= dr\;\alpha_{\mathrm{Imm}}\,\mathrm{e}^{-\alpha_{\mathrm{Imm}}r}\exp-\int_0^r dt\int_{(0,\infty)}\mathrm{e}^{-\ell c(m-t)}\,\ell\nu(d\ell) \\
&= dr\;\alpha_{\mathrm{Imm}}\,\mathrm{e}^{-\int_0^r \phi'(c(m-t))\,dt},
\end{aligned}$$

and, with the notation $\phi_0(\lambda) = \phi(\lambda) - \alpha_{\text{Imm}}\lambda$,

$$\begin{aligned}\mathbb{Q}_m(T_0 \in [r, r+dr], T_0 = T_1) &= \mathbb{Q}_m(T_2 > r)\mathbb{Q}_m(T_1 \in [r, r+dr]) \\ &= dr\ \phi_0'(c(m-r))\,\mathrm{e}^{-\alpha_{\text{Imm}} r} \exp - \int_0^r dt \int_{(0,\infty)} \mathrm{e}^{-\ell c(m-t)}\,\ell\nu(d\ell) \\ &= dr\ \phi_0'(c(m-r))\,\mathrm{e}^{-\int_0^r \phi'(c(m-t))\,dt}.\end{aligned}$$

In particular, we have for $r < m$

$$\mathbb{Q}_m(T_0 \in [r, r+dr]) = dr\ \phi'(c(m-r))\,\mathrm{e}^{-\int_0^r \phi'(c(m-t))\,dt}.$$

and

$$\mathbb{Q}_m(T_0 > r) = \mathrm{e}^{-\int_0^r \phi'(c(m-t))\,dt}. \tag{17}$$

Notice we have $\mathbb{Q}_m(T_0 = \infty) = \exp - \int_0^m \phi'(c(t))\,dt$.

4.1.2. *Conditional law of $\mathcal{N}$ given T_0.* Recall $\mathcal{N}$ is under $\mathbb{Q}_m$ a Poisson point measure with intensity given by (8). Conditionally on $\{T_0 = r, T_0 = T_2\}$, with $m > r > 0$, $\mathcal{N}$ is under $\mathbb{Q}_m$ a point Poisson measure with intensity

$$\mathbf{1}_{[0,r)}(t)\,\mathrm{e}^{-\ell c(m-t)}\,\ell\pi_{\text{Eve}}(d\ell)\delta_0(dz)dt + \\ \mathbf{1}_{(r,m)}(t)\,\mathrm{e}^{-\ell c(m-t)}\,\ell\,[\pi_{\text{Eve}}(d\ell)\delta_0(dz) + \nu(d\ell)\delta_1(dz)]\,dt.$$

Conditionally on $\{T_0 = r, T_0 = T_1\}$, with $r < m$, $\mathcal{N}$ is distributed under $\mathbb{Q}_m$ as $\tilde{\mathcal{N}} + \delta_{(L,r,1)}$ where $\tilde{\mathcal{N}}$ is a point Poisson measure with intensity

$$\mathbf{1}_{[0,r)}(t)\,\mathrm{e}^{-\ell c(m-t)}\,\ell\pi_{\text{Eve}}(d\ell)\delta_0(dz)dt \\ + \mathbf{1}_{(r,m)}(t)\,\mathrm{e}^{-\ell c(m-t)}\,\ell\,[\pi_{\text{Eve}}(d\ell)\delta_0(dz) + \nu(d\ell)\delta_1(dz)]\,dt,$$

and L is a random variable independent of $\tilde{\mathcal{N}}$ with distribution

$$\frac{\mathrm{e}^{-\ell c(m-r)}\,\ell\nu(d\ell)}{\int_{(0,\infty)} \mathrm{e}^{-\ell' c(m-r)}\,\ell'\nu(d\ell')}.$$

Conditionally on $\{T_0 = \infty\}$, $\mathcal{N}$ is under $\mathbb{Q}_m$ a point Poisson measure with intensity

$$\mathbf{1}_{[0,m)}(t)\,\mathrm{e}^{-\ell c(m-t)}\,\ell\pi_{\text{Eve}}(d\ell)\delta_0(dz)dt.$$

4.1.3. *Formulas.* The following two formulas are straightforward: for all $x, \gamma \geq 0$,

$$\psi_{\text{Eve}}'(x+\gamma) - \psi_{\text{Eve}}'(\gamma) = 2\beta x + \int_{(0,\infty)} \mathrm{e}^{-\ell\gamma}\,\ell\pi_{\text{Eve}}(d\ell)[1 - \mathrm{e}^{-\ell x}], \tag{18}$$

$$\psi'(x+\gamma) - \psi'(\gamma) = 2\beta x + \int_{(0,\infty)} \mathrm{e}^{-\ell\gamma}\,\ell\pi(d\ell)[1 - \mathrm{e}^{-\ell x}], \tag{19}$$

Finally we deduce from (5) that $\psi(c) = -c'$, $\psi'(c)c' = -c''$ and

$$\int \psi'(c) = -\log(c'). \tag{20}$$

4.1.4. *Laplace transform.* Recall $\tau_Y = \inf\{t > 0; Y_t = 0\}$ is the extinction time of Y. Let μ_{Eve} and μ_{Total} be two finite measures with support a subset of a finite set $A = \{a_1, \dots, a_n\}$ with $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = \infty$. For $m \in (0, +\infty) \setminus A$, we consider

$$\begin{aligned} w_m(t) &= \mathrm{N}[1 - \mathrm{e}^{-\int Y^0_{r-t}\, \mu_{\text{Eve}}(dr) - \int Y_{r-t}\, \mu_{\text{Total}}(dr)}\, \mathbf{1}_{\{\tau_Y < m-t\}}], \\ w^*_m(t) &= \mathrm{N}[1 - \mathrm{e}^{-\int Y_{r-t}\, \mu_{\text{Total}}(dr)}\, \mathbf{1}_{\{\tau_Y < m-t\}}]. \end{aligned}$$

By noticing that N-a.e. $\mathbf{1}_{\{\tau_Y < m-t\}} = \lim_{\lambda\to\infty} \exp - \int Y_{r-t}\, \mu^\lambda(dr)$, where $\mu^\lambda(dr) = \lambda \delta_m(dr)$, we deduce from Lemma 3.1 in [2] that (w_m, w^*_m) are right continuous and are the unique non-negative solutions of : for $k \in \{0, \dots, n\}$, $m \in (a_k, a_{k+1})$, $t \in (-\infty, m)$,

$$w^*_m(t) + \int_{[t,a_k]} \psi(w^*_m(r))dr = \int_{[t,a_k]} \mu_{\text{Total}}(dr) + c(m - a_k), \tag{21}$$

$$\begin{aligned} w_m(t) + \int_{[t,a_k]} &\psi_{\text{Eve}}(w_m(r))dr \\ &= \int_{[t,a_k]} \mu_{\text{Eve}}(dr) + \int_{[t,a_k]} \mu_{\text{Total}}(dr) + c(m - a_k) + \int_{[t,a_k]} \phi(w^*_m(r))dr. \end{aligned} \tag{22}$$

We define

$$\bar{a}_m = \max\{a_k; a_k < m, k \in \{0, \dots, n\}\}. \tag{23}$$

Notice that $w_m(t) = w^*_m(t) = c(m - t)$ for $t \in (\bar{a}_m, m)$.

4.2. **Proof of Theorem 1.2.**

4.2.1. *Aim.* Theorem 1.2 will be proved as soon as we check that the following equality

$$w(0) = \int_0^\infty -c'(m)\mathbb{Q}_m[1 - \mathrm{e}^{-\int {Y'}^0_r\, \mu_{\text{Eve}}(dr) - \int Y'_r\, \mu_{\text{Total}}(dr)}]dm$$

holds for all the possible choices of measures μ_{Eve} and μ_{Total} satisfying the assumptions of Section 4.1.4, with $w = w_\infty$ defined by (22).

Notice the integrand of the right-hand side is null for $m < a_1$. Let Δ denote the right-hand side. We have for $0 < \varepsilon \le a_1$:

$$\begin{aligned} \Delta &= \int_\varepsilon^\infty dm\ (-c'(m))\mathbb{Q}_m[1 - \mathrm{e}^{-\int {Y'}^0_r\, \mu_{\text{Eve}}(dr) - \int Y'_r\, \mu_{\text{Total}}(dr)}] \\ &= c(\varepsilon) + \int_\varepsilon^\infty dm\ \mathbf{1}_{A^c}(m)c'(m)\mathbb{Q}_m[Z], \end{aligned}$$

with, thanks to the definition (6) of $\kappa_{\max}$,

$$Z = \exp - \int_0^{\bar{a}_m} \kappa_{\max}(dt)\ [n_t \mathbf{1}_{\{t<T_0\}} + n^*_t \mathbf{1}_{\{t\ge T_0\}}]$$

and

$$\begin{aligned} n_t &= \mathrm{N}[(1 - \mathrm{e}^{-\int Y^0_{r-t}\, \mu_{\text{Eve}}(dr) - \int Y_{r-t}\, \mu_{\text{Total}}(dr)})\mathbf{1}_{\{\tau_Y \le m-t\}}] = w_m(t) - c(m - t) \\ n^*_t &= \mathrm{N}[(1 - \mathrm{e}^{-\int Y_{r-t}\, \mu_{\text{Total}}(dr)})\mathbf{1}_{\{\tau_Y \le m-t\}}] = w^*_m(t) - c(m - t), \end{aligned}$$

with (w_m, w^*_m) the non-negative solutions of (21) and (22). Notice that $w_m(t) = w^*_m(t) = c(m - t)$ for $t \in (\bar{a}_m, m)$ and thus $n_t = n^*_t = 0$ when $t \in (\bar{a}_m, m)$.

We set $\Delta = c(\varepsilon) + \displaystyle\int_\varepsilon^\infty \mathbf{1}_{A^c}(m)(\Delta_1+\Delta_2+\Delta_3)\, dm$ with

$$\begin{aligned}
\Delta_1 &= c'(m)\mathbb{Q}_m[Z|T_0 > \bar a_m]\mathbb{Q}_m(T_0 > \bar a_m),\\
\Delta_2 &= c'(m)\int_0^{\bar a_m} \mathbb{Q}_m[Z|T_0 = r, T_0 = T_1]\mathbb{Q}_m(T_0 \in [r, r+dr], T_0 = T_1),\\
\Delta_3 &= c'(m)\int_0^{\bar a_m} \mathbb{Q}_m[Z|T_0 = r, T_0 = T_2]\mathbb{Q}_m(T_0 \in [r, r+dr], T_0 = T_2).
\end{aligned}$$

We shall assume $m \notin A$.

4.2.2. *Computation of* Δ_1. We have, using formula (6),

$$\begin{aligned}
\Delta_1 &= c'(m)\mathbb{Q}_m(T_0 > \bar a_m)\mathbb{Q}_m[\mathrm{e}^{-\int_0^{\bar a_m} \kappa_{\max}(dt) n_t}\, |T_0 > \bar a_m]\\
&= c'(m)\,\mathrm{e}^{-\int_0^{\bar a_m} \phi'(c(m-t))dt}\\
&\quad \exp\left\{-2\beta\int_0^{\bar a_m} (w_m(t) - c(m-t))\; dt - \int_0^{\bar a_m} dt\, \mathrm{e}^{-\ell c(m-t)}\; \ell\pi_{\mathrm{Eve}}(d\ell)[1-\mathrm{e}^{-\ell(w_m(t)-c(m-t))}]\right\}\\
&= c'(m)\,\mathrm{e}^{-\int_0^{\bar a_m} \phi'(c(m-t))dt} \exp\left\{-\int_0^{\bar a_m} dt[\psi'_{\mathrm{Eve}}(w_m(t)) - \psi'_{\mathrm{Eve}}(c(m-t))]\right\}\\
&= c'(m)\,\mathrm{e}^{\int_{m-\bar a_m}^m \psi'(c(t))dt}\, \mathrm{e}^{-\int_0^{\bar a_m} dt\; \psi'_{\mathrm{Eve}}(w_m(t))}\\
&= c'(m-\bar a_m)\,\mathrm{e}^{-\int_0^{\bar a_m} dt\; \psi'_{\mathrm{Eve}}(w_m(t))},
\end{aligned}$$

where we used (20) for the last equality to get

$$\mathrm{e}^{\int_{m-\bar a_m}^m \psi'(c(t))dt} = \mathrm{e}^{-[\log(c'(t))]_{m-\bar a_m}^m} = \frac{c'(m-\bar a_m)}{c'(m)}. \tag{24}$$

4.2.3. *Computation of* Δ_2. Using Section 4.1.2, we get

$$\begin{aligned}
\mathbb{Q}_m[Z|T_0 = r, T_0 = T_1] &= \mathrm{e}^{-2\beta\int_0^r (w_m(t)-c(m-t))\; dt - 2\beta \int_r^{\bar a_m}(w_m^*(t)-c(m-t))\; dt}\\
&\qquad \exp(-\int_0^r dt\,\mathrm{e}^{-\ell c(m-t)}\;\ell\pi_{\mathrm{Eve}}(d\ell)[1-\mathrm{e}^{-\ell n_t}])\\
&\qquad \exp(-\int_r^{\bar a_m} dt\,\mathrm{e}^{-\ell c(m-t)}\;\ell\pi(d\ell)[1-\mathrm{e}^{-\ell n_t^*}])\\
&\qquad \int_{(0,\infty)} \nu(d\ell')\frac{\mathrm{e}^{-\ell' c(m-r)}\;\ell'\,\mathrm{e}^{-\ell' n_r^*}}{\phi_0'(c(m-r))}\\
&= \exp(-\int_0^r dt[\psi'_{\mathrm{Eve}}(w_m(t)) - \psi'_{\mathrm{Eve}}(c(m-t))])\\
&\qquad \exp(-\int_r^{\bar a_m} dt[\psi'(w_m^*(t)) - \psi'(c(m-t))])\\
&\qquad \frac{\phi_0'(w_m^*(r))}{\phi_0'(c(m-r))}.
\end{aligned}$$

We deduce from Section 4.1.1

$$\begin{aligned}
\Delta_2 &= c'(m)\int_0^{\bar a_m} \mathbb{Q}_m[Z|T_0=r, T_0=T_1]\mathbb{Q}_m(T_0\in[r,r+dr], T_0=T_1),\\
&= c'(m)\int_0^{\bar a_m} dr\ \phi_0'(w_m^*(r))\,\mathrm{e}^{-\int_0^r \phi'(c(m-t))\,dt}\exp(-\int_0^r dt[\psi'_{\mathrm{Eve}}(w_m(t))-\psi'_{\mathrm{Eve}}(c(m-t))])\\
&\qquad \exp(-\int_r^{\bar a_m} dt[\psi'(w_m^*(t))-\psi'(c(m-t))])\\
&= c'(m)\,\mathrm{e}^{\int_0^{\bar a_m}\psi'(c(m-t))\,dt}\int_0^{\bar a_m} dr\ \phi_0'(w_m^*(r))\,\mathrm{e}^{-\int_0^r dt\,\psi'_{\mathrm{Eve}}(w_m(t))-\int_r^{\bar a_m} dt\,\psi'(w_m^*(t))}\\
&= c'(m-\bar a_m)\int_0^{\bar a_m} dr\ \phi_0'(w_m^*(r))\,\mathrm{e}^{-\int_0^r dt\,\psi'_{\mathrm{Eve}}(w_m(t))-\int_r^{\bar a_m} dt\,\psi'(w_m^*(t))},
\end{aligned}$$

where we used (24) for the last equality.

4.2.4. *Computation of* Δ_3. Using Section 4.1.2, we get

$$\begin{aligned}
\mathbb{Q}_m[Z|T_0=r,T_0=T_2] &= \mathrm{e}^{-2\beta\int_0^r (w_m(t)-c(m-t))\,dt-2\beta\int_r^{\bar a_m}(w_m^*(t)-c(m-t))\,dt}\\
&\qquad \exp\left\{-\int_0^r dt\,\mathrm{e}^{-\ell c(m-t)}\,\ell\pi_{\mathrm{Eve}}(d\ell)[1-\mathrm{e}^{-\ell n_t}]\right\}\\
&\qquad \exp\left\{-\int_r^{\bar a_m} dt\,\mathrm{e}^{-\ell c(m-t)}\,\ell\pi(d\ell)[1-\mathrm{e}^{-\ell n_t^*}]\right\}\\
&= \exp(-\int_0^r dt[\psi'_{\mathrm{Eve}}(w_m(t))-\psi'_{\mathrm{Eve}}(c(m-t))])\\
&\qquad \exp(-\int_r^{\bar a_m} dt[\psi'(w_m^*(t))-\psi'(c(m-t))]).
\end{aligned}$$

We deduce from Section 4.1.1

$$\begin{aligned}
\Delta_3 &= c'(m)\int_0^{\bar a_m}\mathbb{Q}_m[Z|T_0=r,T_0=T_2]\mathbb{Q}_m(T_0\in[r,r+dr],T_0=T_2),\\
&= c'(m)\int_0^{\bar a_m} dr\ \alpha_{\mathrm{Imm}}\,\mathrm{e}^{-\int_0^r \phi'(c(m-t))\,dt}\exp(-\int_0^r dt[\psi'_{\mathrm{Eve}}(w_m(t))-\psi'_{\mathrm{Eve}}(c(m-t))])\\
&\qquad \exp(-\int_r^{\bar a_m} dt[\psi'(w_m^*(t))-\psi'(c(m-t))])\\
&= c'(m)\,\mathrm{e}^{\int_0^{\bar a_m}\psi'(c(m-t))\,dt}\int_0^{\bar a_m} dr\ \alpha_{\mathrm{Imm}}\,\mathrm{e}^{-\int_0^r dt\,\psi'_{\mathrm{Eve}}(w_m(t))-\int_r^{\bar a_m} dt\,\psi'(w_m^*(t))}\\
&= c'(m-\bar a_m)\int_0^{\bar a_m} dr\ \alpha_{\mathrm{Imm}}\,\mathrm{e}^{-\int_0^r dt\,\psi'_{\mathrm{Eve}}(w_m(t))-\int_r^{\bar a_m} dt\,\psi'(w_m^*(t))},
\end{aligned}$$

where we used (24) for the last equality.

4.2.5. *Computation of* $\Delta_2+\Delta_3$. We have

$$\Delta_2+\Delta_3 = c'(m-\bar a_m)\int_0^{\bar a_m} dr\ \phi'(w_m^*(r))\,\mathrm{e}^{-\int_0^r dt\,\psi'_{\mathrm{Eve}}(w_m(t))-\int_r^{\bar a_m} dt\,\psi'(w_m^*(t))}\,.$$

Differentiating (21) w.r.t. time and m, we get for $t<m$

$$\partial_m(w_m^*)'(t)-\partial_m w_m^*(t)\psi'(w_m^*(t))=0.$$

Notice also that for $m > t \geq \bar{a}_m$, we have $\partial_m w^*(t) = c'(m-t)$ and thus

$$\partial_m w^*(\bar{a}_m) = c'(m - \bar{a}_m).$$

We get

$$\exp(-\int_r^{\bar{a}_m} dt\psi'(w_m^*(t))) = \frac{\partial_m w_m^*(r)}{\partial_m w_m^*(\bar{a}_m)} = \frac{\partial_m w_m^*(r)}{c'(m-\bar{a}_m)}.$$

Differentiating (22) w.r.t. time and m, we get for $t < m$

$$\partial_m w_m'(t) - \partial_m w_m(t)\psi_{\mathrm{Eve}}'(w_m(t)) = -\partial_m w_m^*(t)\phi'(w_m^*(t)).$$

We deduce that

$$\begin{aligned}
\Delta_2 + \Delta_3 &= \int_0^{\bar{a}_m} dr \partial_m w_m^*(t)\phi'(w_m^*(r))\, \mathrm{e}^{-\int_0^r dt\ \psi_{\mathrm{Eve}}'(w_m(t))} \\
&= -\int_0^{\bar{a}_m} dr[\partial_m w_m'(r) - \partial_m w_m(r)\psi_{\mathrm{Eve}}'(w_m(r))]\, \mathrm{e}^{-\int_0^r dt\ \psi_{\mathrm{Eve}}'(w_m(t))} \\
&= -\left[\partial_m w_m(r)\, \mathrm{e}^{-\int_0^r dt\ \psi_{\mathrm{Eve}}'(w_m(t))}\right]_0^{\bar{a}_m} \\
&= \partial_m w_m(0) - \partial_m w_m(\bar{a}_m)\, \mathrm{e}^{-\int_0^{\bar{a}_m} dt\ \psi_{\mathrm{Eve}}'(w_m(t))}.
\end{aligned}$$

Notice also that for $m > t \geq \bar{a}_m$ one has $\partial_m w(t) = c'(m-t)$, in particular $\partial_m w(\bar{a}_m) = c'(m - \bar{a}_m)$. This implies that

$$\Delta_2 + \Delta_3 = \partial_m w_m(0) - c'(m - \bar{a}_m)\, \mathrm{e}^{-\int_0^{\bar{a}_m} dt\ \psi_{\mathrm{Eve}}'(w_m(t))}.$$

4.3. **Conclusion.** Thus, for $m \notin A$, we have

$$\Delta_1 + \Delta_2 + \Delta_3 = \partial_m w_m(0),$$

and

$$\Delta = c(\varepsilon) + \int_\varepsilon^\infty \partial_m w_m(0) = c(\varepsilon) + w_\infty(0) - w_\varepsilon(0) = w(0).$$

This ends the proof of the Theorem.

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